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A decomposition solution for fins with temperature dependent surface heat flux

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Abstract

In this study, the Adomian decomposition method is used to analyze the thermal characteristics of a straight rectangular fin for all possible types of heat transfer. The local heat transfer coefficient is assumed to vary with a power-law function of temperature. Instead of a traditionally implicit form of solution, the decomposition solution gives an explicit expression of temperature distribution as a function of position along the fin. The obtained decomposed analytic solution is in the form of an infinite power series and the series can be truncated in a practical way to obtain numerical results. Thus, the fin tip temperature, fin base heat transfer rate, and fin efficiency can be calculated directly without the need of iteration. Results indicate that the series converges rapidly with high accuracy and seems to be convenient for the engineering application.

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1. Introduction

Fins are frequently used in engineering to enhance the rate of heat transfer on a solid surface. For the cases of constant heat transfer coefficient, the analytical solutions of temperature distribution as well as heat transfer rate can be easily obtained [1]. However, in some situations such as fins in boiling liquids, the heat transfer coefficient is no longer uniform and varies with the temperature difference between the surface and the adjacent fluid in a nonlinear manner. The dependence of the heat transfer coefficient on the local temperature difference can be governed by a power-law-type form. Consequently, the equation for temperature becomes highly nonlinear and is difficult to obtain an analytical solution.

Numerous studies have devoted to the analysis of fin performance of this type of problems due to its important application in engineering. Lai and Hsu [2] assumed a simple model to calculate the base heat flux and the length of the nucleate boiling section. Their results were compared consistently with those of Haley and Westwater [3]. Mehta and Aris [4,5] considered the same equation related to the problem of diffusion and reaction in a porous slab. They gave the solutions for all orders of reaction in terms of the hyper-geometric function. Later Ünal [6–9] made a series studies on an extended surface with nonuniform heat transfer coefficient and showed that the equation can be integrated analytically in a closed form for a limited number of cases. Sen and Trinh [10] used the model of Mehta and Aris to compare the heat transfer rate of a single fin cooled by natural convection, nucleate boiling, and radiation. Liaw and Yeh [11] used the same model and further studied all possible types of heat transfer including the cases of film

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Nomenclature

A	cross-section area of the fin
A_n	Adomian's polynomial
a	dimensional constant in Eq. (1)
C	integral constant
h	local heat transfer coefficient
k	thermal conductivity of the fin
L	the length of the fin
L_X	operator of the highest order of derivative
L_X^{-1}	inverse operator of L_X
N	dimensionless fin parameter defined in Eq. (3)
n	exponent in Eq. (1)
P	periphery of the fin cross-section area
Q_b	dimensionless temperature gradient at fin base
T	temperature
X	nondimensional space coordinate
x	dimensional space coordinate

Greek symbols

θ	dimensionless temperature of the fin
φ_m	m-terms summation of Adomian's polynomial
η	fin efficiency

Subscripts

a	refer to the ambient property
b	refer to the fin base
e	refer to the fin tip
m	number of terms in the series

and transition boiling with and without heat transfer at fin tip. They also conducted an analytical and experimental study for a fin with various types of boiling occurring simultaneously at adjacent locations on its surface [12]. However, the hyper-geometric function and Dawson's integral [11] are implicit expressions and can be solved only by iterations to find the fin tip temperature as well as the temperature distribution along the fin. Dul'kin and Garas'ko [13,14] pointed out that such calculations are acceptable in scientific investigation but are inconvenient for engineering application. Thus, they derived a closed form solution for this problem by uses of the quotient and fitting procedure for the exact hyper-geometric and well-known hyperbolic solutions. The obtained formula in terms of ordinary functions gives an expression for direct evaluation of fin tip temperature in a wide range of assigned values of exponent of the power-law function and fin parameter. The heat transfer rate at fin base and fin effectiveness can also be determined [14]. Their results were compared consistently with those obtained by numerical integration.

Recently, the Adomian decomposition method has been used to solve a wide range of physical problems [15–21]. This method provides a direct scheme for solving linear and nonlinear deterministic and stochastic equations without the need for linearization and yields rapidly convergent series solutions. Chiu and Chen [22] have applied this method to analyze the performance of longitudinal fin with constant heat transfer coefficient and variable thermal conductivity. Their results showed that the decomposition solution has many merits including fast convergence and high accuracy. The objective of this study is to apply the Adomian decomposition method to investigate a straight fin governed by a power-law-type temperature dependent heat transfer coefficient. Based on the decomposed analytical solution, the temperature on the fin surface can be expressed explicitly as a function of position along the fin. The effects of exponent value and fin parameter on the temperature distribution as well as the fin tip temperature can also be obtained quickly. In addition, the heat transfer rate at fin base and fin efficiency are presented in detail. Results are compared with those of [13,14].

2. Problem formulation and decomposition method

Consider a straight fin of length L with a uniform cross-section area A. The fin surface is exposed to a convective environment at temperature T_a and the local heat transfer coefficient h along the fin surface is assumed to exhibit a power-law-type dependence on the local temperature difference between the fin and the ambient fluid as

$$h = a(T - T_a)^n, \tag{1}$$

where a is a dimensional constant defined by physical properties of the surrounding medium, T is the local temperature on the fin surface, and the exponent n depends on the heat transfer mode. The value of n can vary in a wide range between −4 and 5 [11,12]. For example, the exponent n may take the values −4, −0.25, 0, 2, and 3, indicating the fin subject to transition boiling, laminar film boiling or condensation, convection, nucleate boiling, and radiation into free space at zero absolute temperature, respectively. For one-dimensional steady state heat conduction, the equation in terms of dimensionless variables $X = x/L$ and $\theta = (T - T_a)/(T_b - T_a)$ can be written as

$$\frac{d^2\theta}{dX^2} - N^2\theta^{n+1} = 0, \tag{2}$$

where the axial distance X is measured from the fin tip, T_b is the fin base temperature, and N is the convective–conductive parameter of the fin defined as

$$N = \left(\frac{h_b PL^2}{kA}\right)^{\frac{1}{2}} = \left[\frac{aPL^2}{kA}(T_b - T_a)^n\right]^{\frac{1}{2}}. \tag{3}$$

In the above equation h_b, P, and k represent the heat transfer coefficient at fin base, the periphery of fin cross-section, and the conductivity of the fin, respectively. For simplicity, assume the fin tip is insulated and the boundary conditions to Eq. (2) can be expressed as

$$X = 0, \quad \frac{d\theta}{dX} = 0, \tag{4}$$

$$X = 1, \quad \theta = 1. \tag{5}$$

According to the Adomian decomposition method [21], we can define the linear operator $L_X = d^2/dX^2$. Consequently, Eq. (2) becomes

$$L_X\theta = N^2\theta^{n+1} = N^2(NA), \tag{6}$$

where $NA = \theta^{n+1}$ represents the nonlinear term. The decomposition technique expands the solution of θ in a series form

$$\theta = \sum_{i=0}^{\infty}\theta_i \tag{7}$$

and the nonlinear term can be decomposed by Adomian's polynomials [21] in the following form

$$NA = \sum_{i=0}^{\infty}A_i, \tag{8}$$

where A_i can be obtained by the formula

$$A_0 = f(\theta_0) \quad \text{and} \quad A_i = \sum_{v=1}^{i} c(v,i)f^{(v)}(\theta_0), \quad i \geqslant 1. \tag{9}$$

In the above equation, $c(v,i)$ are products of the v components of θ whose subscripts sum to i, divided by the factorial of the number of repeated subscripts. Accordingly, the A_i's are expressed as

$$\begin{aligned}
A_0 &= f(\theta_0) = \theta_0^{n+1},\\
A_1 &= c(1,1)f^{(1)}(\theta_0) = \theta_1\frac{df(\theta_0)}{d\theta_0} = (n+1)\theta_1\theta_0^n,\\
A_2 &= c(1,2)f^{(1)}(\theta_0) + c(2,2)f^{(2)}(\theta_0)\\
&= \theta_2\frac{df(\theta_0)}{d\theta_0} + \frac{\theta_1^2}{2!}\frac{d^2f(\theta_0)}{d\theta_0^2}\\
&= (n+1)\theta_2\theta_0^n + \frac{1}{2}n(n+1)\theta_1^2\theta_0^{n-1},\\
A_3 &= c(1,3)f^{(1)}(\theta_0) + c(2,3)f^{(2)}(\theta_0)\\
&\quad + c(3,3)f^{(3)}(\theta_0)\\
&= \theta_3\frac{df(\theta_0)}{d\theta_0} + \theta_1\theta_2\frac{d^2f(\theta_0)}{d\theta_0^2} + \frac{\theta_1^3}{3!}\frac{d^3f(\theta_0)}{d\theta_0^3}\\
&= (n+1)\theta_3\theta_0^n + n(n+1)\theta_1\theta_2\theta_0^{n-1}\\
&\quad + \frac{1}{6}n(n+1)(n-1)\theta_1^3\theta_0^{n-2}, \cdots
\end{aligned} \tag{10}$$

So far, θ_i's can be assigned arbitrarily. However, they are taken in the following way for optimal convergence. We impose the inverse operator L_X^{-1} on both sides of Eq. (6) yields

$$\sum_{i=0}^{\infty}\theta_i = \theta_0 + N^2L_X^{-1}\sum_{i=0}^{\infty}A_i, \tag{11}$$

where θ_0 is the one term approximation of θ and can be assigned as

$$\theta_0 = \theta(0) + X\frac{d\theta(0)}{dX}. \tag{12}$$

From the boundary condition (4) and taking $\theta(0)$ be an arbitrary constant C, the decomposition solution can be obtained by the recursive relationship derived from Eq. (11)

$$\theta_{i+1} = N^2L_X^{-1}A_i, \quad i \geqslant 0. \tag{13}$$

Thus, all components of θ are determinable since A_0 depends only on θ_0, and then A_1 depends on θ_0 and θ_1, etc. The first four iterates are expressed as the following:

$$\begin{aligned}
\theta_0 &= C,\\
\theta_1 &= \frac{1}{2}C^{1+n}N^2X^2,\\
\theta_2 &= \frac{1}{24}(1+n)C^{1+2n}N^4X^4,\\
\theta_3 &= \frac{1}{720}(1+5n+4n^2)C^{1+3n}N^6X^6,\\
&\cdots
\end{aligned} \tag{14}$$

The practical solution will be the m-terms approximation φ_m to θ, which is usually written as

$$\varphi_m = \sum_{i=0}^{m-1}\theta_i = \theta_0 + \theta_1 + \theta_2 + \cdots + \theta_{m-1}, \quad m \geqslant 1, \tag{15}$$

so that φ_m approaches θ as $m \to \infty$. The other boundary condition given by Eq. (5) is used to evaluate the constant C. Note that the value of C must lie in the interval $(0,1)$ to represent the temperature at the fin tip. Once the value of C is determined, Eq. (15) gives an approximation analytical solution of θ for assigned values of parameters N and n, and the temperature distribution along the fin is expressed explicitly as a function of position X. Moreover, the analysis of fin performance can be easily performed by the use of this equation.

3. Results and discussion

First we examine the convergence of the present method by defining $\Delta\varphi = \varphi_{m+1}(0) - \varphi_m(0)$ which expressed the derivation as one term is added to the solution at $X = 0$. Fig. 1 shows the variations of $\Delta\varphi$ with the number of components m for the case $N = 1$ with several assigned values of n. Note that there is no solution for integer values of n less than -1 at $N = 1$. It is found that the deviation $\Delta\varphi$ reduces rapidly with m and is less than the order of magnitude 10^{-4} as $m > 5$ for all the cases considered, which indicates the convergence of the series is quite excellent. Several cases with variations of parameters N and n are also tested and it can be calculated that the use of 13 terms in Eq. (15) is sufficient to yield accurate results.

The relationship between N and θ_e for $-4 \leqslant n \leqslant 5$ in the usually used range $0 < N \leqslant 5$ is determined in Fig. 2, where $\theta_e = \theta(0)$ represents the fin tip temperature. It is found that the present results are in excellent agreement with those obtained by [13] with numerically integration method. The dashed curves corresponding to each value of n are the results of [13] from the derived close-form relationship for the approximately dependence of the fin parameter N on θ_e and n. Obviously, the present method gives fast and accurate results instead of complicated numerical integration and iteration procedure. It is noted that each curve for $n < -1$ has a peak value. Sen and Trinh [10] have proved that only the branch on the right-hand side of the peak corresponds to the physically stable and realizable states by means of the linear stability analysis.

The temperature profiles for several assigned values of n at $N = 1$ are displayed in Fig. 3. As indicated in Eq. (15), the temperature along the fin is expressed in an explicit function of position X. Thus, the temperature profile can be easily obtained for any exponent value n. The characteristics of temperature profiles have been discussed by Liaw and Yeh [11] and Dul'kin and Garas'ko [14]. The former used the hyper-geometric formulas to determine the profiles and the latter derived an inversed form for the temperature distribution along

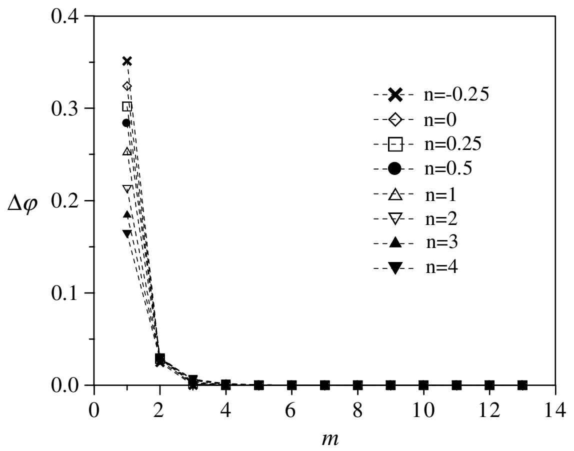

Fig. 1. The variations of $\Delta\varphi$ with m for several assigned values of n with $N = 1$.

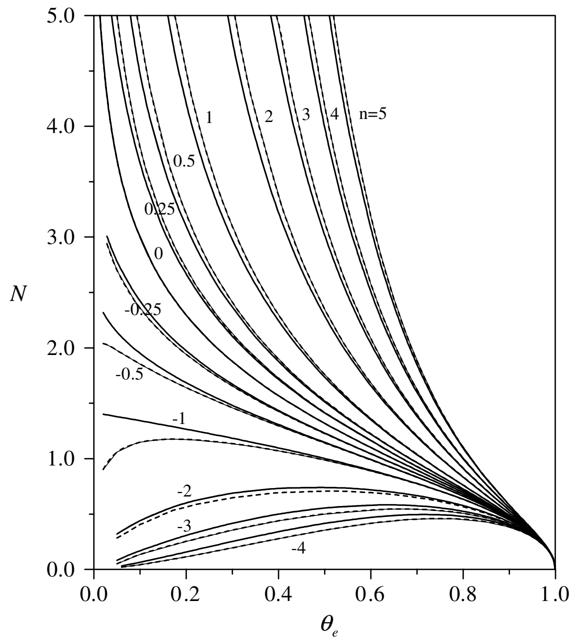

Fig. 2. The variational relationships of N and θ_e for several assigned values of n. The solid lines are the present results and the dashed lines are from the approximate formula of [13].

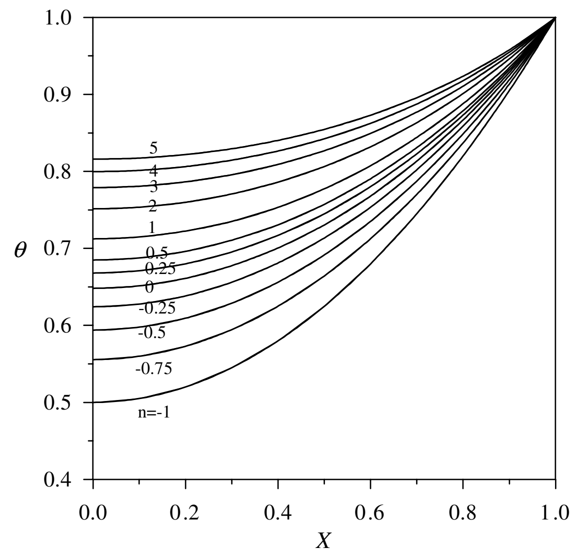

Fig. 3. The temperature profiles for several assigned values of n at $N = 1$.

the fin, and then evaluated the profile via an iterative procedure. The present results are consistent with both of them while with more straightforward process.

The amount of the energy transferred from the fin base is of great interest in engineering and can be indicated by the dimensionless temperature gradient Q_b with the definition $Q_b = d\theta(1)/dX$. Fig. 4 illustrates the dependence of Q_b and N for some typical values of n. The results can be simply obtained via the direct differentiation of Eq. (15). It is found that the parameter Q_b increases with N for all the values n considered. Note that only the physically stable states of the curve are shown for the cases $n = -2$, -3 and -4 in this figure.

The other properties such as the fin efficiency as well as the fin effectiveness also can be easily determined. For instance, if we define the fin efficiency η in the usual way as the ratio of total heat transfer rate to that of fin at base temperature, we get

$$\eta = \frac{\int_0^L Ph(T - T_a)\,dx}{PLh_b(T_b - T_a)} = \int_0^1 \theta^{n+1}\,dX. \tag{16}$$

The results are demonstrated in Fig. 5 for several values of n. Note that the case $n = -1$ indicates an uniform local heat flux over the whole fin surface and induces the result of $\eta = 1$. So the Eq. (16) is valid only for the cases $n \geqslant -1$ as shown in Fig. 5. It is found that the value of η decreases with N as $n > -1$, and the decreasing rate increases with the exponent value of n.

4. Conclusions

The Adomian decomposition method has been used to analyze the heat conduction problem for a fin with

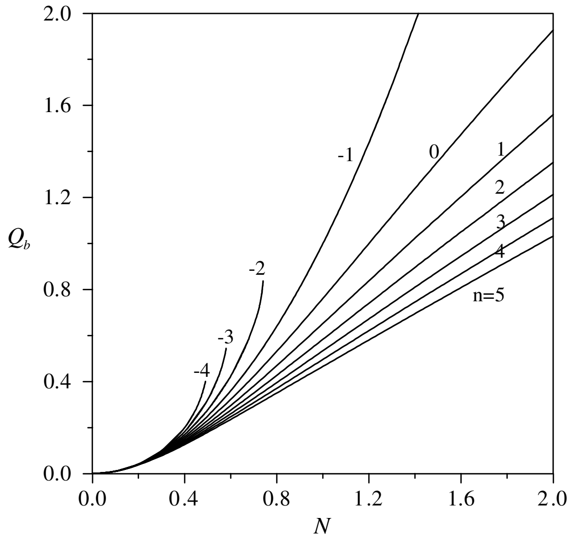

Fig. 4. The variations of Q_b with N for several assigned values of n.

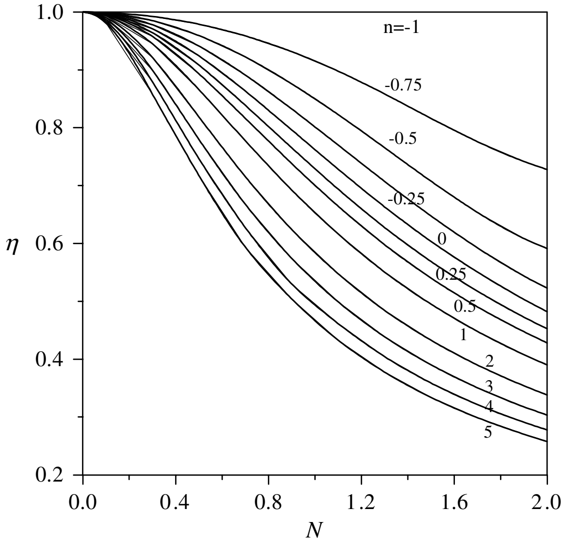

Fig. 5. The variations of fin efficiency η with N for several assigned values of n.

heat transfer coefficient varying as a power-law function of temperature. This method provides a simply approximate exact solution without any assumption of linearization. This character is very important for systems with strong nonlinearities which could be extremely sensitive to small changes in parameters. The obtained solution for temperature distribution offers many advantages over the other methods including fast convergence and high accuracy, and especially gives an explicit form of solution. Thus, the other important properties of fin such as the fin base thermal conductance and fin efficiency can be quickly evaluated from the explicit solution. It would be useful to apply this method to a variety of nonlinear heat conduction problems, and helpful for engineer to analyze highly nonlinear system.

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